

ON THE TODA AND KAC-VAN MOERBEKE SYSTEMS

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ABSTRACT. Given a solution of the Toda lattice we explicitly construct a solution of the Kac-van Moerbeke system related to each other by a Miura-type transformation. As an illustration of our method we derive the N -soliton solutions of the Kac-van Moerbeke lattice.

1. INTRODUCTION

We extend our previous work [11, 12] on connections between the Korteweg-deVries and modified Korteweg-deVries equation based on Miura's transformation [23] to the Toda lattice and its modified version, the Kac-van Moerbeke system. The Toda and Kac-van Moerbeke equations, in appropriate variables [7], are defined by [19, 22, 26, 29]

$$(1.1) \quad \begin{aligned} \dot{a}(t, n) &= a(t, n)[b(t, n) - b(t, n+1)], \\ \dot{b}(t, n) &= 2[a(t, n-1)^2 - a(t, n)^2], \end{aligned} \quad (t, n) \in \mathbb{R} \times \mathbb{Z},$$

and

$$(1.2) \quad \dot{\alpha}(t, n) = \alpha(t, n)[\alpha(t, n+1)^2 - \alpha(t, n-1)^2], \quad (t, n) \in \mathbb{R} \times \mathbb{Z},$$

respectively. Here “ $\dot{}$ ” denotes d/dt , a, b, α are real-valued sequences, and $(a+b)(n) := a(n) + b(n)$, $(ab)(n) := a(n)b(n)$, $n \in \mathbb{Z}$. The analog of Miura's transformation between the two nonlinear lattices now reads as follows. Let α be a solution of the Kac-van Moerbeke system (1.2) and define

$$(1.3) \quad \begin{aligned} a_1(t, n) &:= \alpha(t, 2n)\alpha(t, 2n+1), \\ a_2(t, n) &:= \alpha(t, 2n+1)\alpha(t, 2n+2), \\ b_1(t, n) &:= -\alpha(t, 2n-1)^2 - \alpha(t, 2n)^2, \\ b_2(t, n) &:= -\alpha(t, 2n)^2 - \alpha(t, 2n+1)^2, \end{aligned} \quad (t, n) \in \mathbb{R} \times \mathbb{Z}.$$

Then (a_1, b_1) and (a_2, b_2) solve the Toda lattice equations (1.1). Our main objective in this paper is to reverse this process, i.e., starting with a solution, say (a_1, b_1) of (1.1) we shall construct a solution α of (1.2) and another solution (a_2, b_2) of (1.1) that are linked to each other by the Miura-type transformations (1.3).

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The transformations (1.3), according to a footnote in [24], seem to go back to M. Hénon. They have been used in [24] in connection with the scattering problem for the finite Kac-van Moerbeke system [19]. The corresponding semi-infinite system has been studied in [19] (see also [31]). The Kac-van Moerbeke system (1.2) on \mathbb{Z} is treated, e.g., in [22, 26, 29] and, due to its applications in modeling Langmuir oscillations in plasmas, is also called the Langmuir lattice. For studies of the $2N$ -periodic system (1.2) (i.e., $\alpha(t, n + 2N) = \alpha(t, n)$) we refer, e.g., to [1, 20, 21, 32].

In §2 we present our main arguments concerning the construction of solutions α of (1.2) given a solution (a_1, b_1) of (1.1) related to each other via (1.3). As a technical tool we employ certain results on positive, weak solutions of second order finite difference operators which are summarized in Appendix A. In §3 we transfer the class of N -soliton solutions of the Toda lattice (1.1) to the Kac-van Moerbeke system (1.2).

Finally we mention that our methods are not confined to $(1+1)$ -dimensional systems such as the (modified) Korteweg-deVries equation, Gelfand-Dikii and Drinfeld-Sokolov hierarchies, and Toda systems, but also apply to the (modified) Kadomtsev-Petviashvili equation [9].

2. CONNECTIONS BETWEEN THE TODA AND KAC-VAN MOERBEKE SYSTEMS

In this section we extend our previous treatment of the Korteweg-deVries and modified Korteweg-deVries equations [11], [12], to the Toda lattice and its modified version, the Kac-van Moerbeke system.

In $l^\infty(\mathbb{Z})$ we introduce shift operators S^\pm by

$$(2.1) \quad (S^\pm f)(n) := f(n \pm 1), \quad n \in \mathbb{Z}, \quad f = \{f(n)\}_{n \in \mathbb{Z}} \in l^\infty(\mathbb{Z}),$$

and also use the notation

$$(2.2) \quad f^\pm := S^\pm f.$$

Assuming hypothesis

$$(H.2.1) \quad a(t) = \{a(t, n)\}_{n \in \mathbb{Z}} \in l^\infty_{\mathbb{R}}(\mathbb{Z}), \quad b(t) = \{b(t, n)\}_{n \in \mathbb{Z}} \in l^\infty_{\mathbb{R}}(\mathbb{Z}), \\ t \in \mathbb{R}, \quad a(\cdot, n), b(\cdot, n) \in C^\infty(\mathbb{R}), \quad n \in \mathbb{Z},$$

the Toda lattice (TL), in Flaschka's variables [7], reads

$$(2.3) \quad TL(a, b) := (\dot{a} - a[b - b^+], \dot{b} - 2[(a^-)^2 - a^2]) = 0,$$

where “ $\dot{}$ ” denotes d/dt and $l^\infty_{\mathbb{R}}(\mathbb{Z})$ denotes the algebra of real-valued sequences in $l^\infty(\mathbb{Z})$.

Assuming hypothesis

$$(H.2.2) \quad \alpha(t) = \{\alpha(t, n)\}_{n \in \mathbb{Z}} \in l^\infty_{\mathbb{R}}(\mathbb{Z}), \quad t \in \mathbb{R}, \quad \alpha(\cdot, n) \in C^\infty(\mathbb{R}), \quad n \in \mathbb{Z},$$

the corresponding modified Toda lattice, the Kac-van Moerbeke (KM) lattice is then defined by

$$(2.4) \quad KM(\alpha) := \dot{\alpha} - \alpha[(\alpha^+)^2 - (\alpha^-)^2] = 0.$$

The connection between (2.3) and (2.4) is the following: Suppose α satisfies (H.2.2) and (2.4) and define

$$(2.5) \quad \phi(t, n) := \alpha(t, 2n), \quad \psi(t, n) := \alpha(t, 2n + 1), \quad (t, n) \in \mathbb{R} \times \mathbb{Z},$$

and

$$(2.6) \quad a_1(t) := \phi(t)\psi(t), \quad b_1(t) := -\phi(t)^2 - \psi^-(t)^2, \quad t \in \mathbb{R},$$

$$(2.7) \quad a_2(t) := \phi^+(t)\psi(t), \quad b_2(t) := -\phi(t)^2 - \psi(t)^2, \quad t \in \mathbb{R}.$$

Then (a_j, b_j) , $j = 1, 2$, satisfy (H.2.1) and (2.3). In particular,

$$(2.8) \quad \text{KM}(\alpha) = 0 \Rightarrow \text{TL}(a_j, b_j) = 0, \quad j = 1, 2$$

[19, 24] (see also [1, 26, 29]). Thus (2.6), (2.7) play a role analogous to Miura's transformation [23] in the (modified) Korteweg-deVries context. We also note that the transform from (a_1, b_1) to (a_2, b_2) is simply given by the replacement

$$(2.9) \quad \alpha \rightarrow S^+ \alpha$$

in (1.3). Our main objective in this paper is to reverse the implication in (2.8), i.e., given a solution (a_1, b_1) of (2.3), construct solutions α of (2.4) and (a_2, b_2) of (2.3) related to each other by (2.6)–(2.8).

We start by recalling the Lax pairs associated with the TL and KM equations and refer to [5] for relevant Hilbert space methods. Assume (H.2.1) and define on $l^2(\mathbb{Z})$

$$(2.10) \quad H(t) := a(t)S^+ + a^-(t)S^- - b(t), \quad t \in \mathbb{R},$$

$$(2.11) \quad B_H(t) := a(t)S^+ - a^-(t)S^-, \quad t \in \mathbb{R},$$

$$(2.12) \quad A(t) := \psi(t)S^+ + \phi(t), \quad A(t)^* = \psi^-(t)S^- + \phi(t), \quad t \in \mathbb{R},$$

and on $l^2(\mathbb{Z}) \otimes \mathbb{C}^2$

$$(2.13) \quad Q(t) := \begin{pmatrix} 0 & A(t)^* \\ A(t) & 0 \end{pmatrix}, \quad t \in \mathbb{R},$$

$$(2.14) \quad B_Q(t) := \begin{pmatrix} B_{A^*A}(t) & 0 \\ 0 & B_{AA^*}(t) \end{pmatrix} = B_{A^*A}(t) \oplus B_{AA^*}(t), \quad t \in \mathbb{R},$$

where ϕ and ψ are defined as in (2.5). Then

$$(2.15) \quad \dot{H} - [B_H, H] = 0 \quad \text{iff} \quad \text{TL}(a, b) = 0$$

and

$$(2.16) \quad \dot{Q} - [B_Q, Q] = 0 \quad \text{iff} \quad \text{KM}(\alpha) = 0$$

implying

Theorem 2.3 [7, 19] (see also [24, 26, 29]). (i) Suppose (a, b) satisfies (H.2.1) and $\text{TL}(a, b) = 0$. Then there exists a family of unitary operators $U(t)$, $t \in \mathbb{R}$, $U(0) = 1$ in $l^2(\mathbb{Z})$ such that

$$(2.17) \quad U(t)^{-1}H(t)U(t) = H(0), \quad t \in \mathbb{R}.$$

(ii) Suppose α satisfies (H.2.2) and $\text{KM}(\alpha) = 0$. Then there exists a family of unitary operators $W(t)$, $t \in \mathbb{R}$, $W(0) = 1$ in $l^2(\mathbb{Z}) \otimes \mathbb{C}^2$ such that

$$(2.18) \quad W(t)^{-1}Q(t)W(t) = Q(0), \quad t \in \mathbb{R}.$$

Clearly this theorem extends to unbounded sequences a, b, α as long as appropriate domain considerations for $H(t)$ and $Q(t)$ are taken into account.

Next we recall the definition of the Wronskian determinant associated with (2.10),

$$(2.19) \quad W(f, g)(n) := a(n) [f(n)g(n+1) - f(n+1)g(n)],$$

and define weak solutions f of $Hf = zf$, $z \in \mathbb{C}$, by

$$(2.20) \quad ((H - \bar{z})g, f) := \sum_{n \in \text{supp}(g)} \overline{((H - \bar{z})g)(n)} f(n) = 0, \quad g \in l_0(\mathbb{Z}),$$

where $l_0(\mathbb{Z})$ denotes the set of complex-valued sequences $g = \{g(n)\}_{n \in \mathbb{Z}}$ with only finitely many $g(n) \neq 0$.

Lemma 2.4. *Suppose (a, b) satisfies (H.2.1) and $\text{TL}(a, b) = 0$ and let $f_0(\lambda)$ be a real-valued weak solution of $H(0)f_0(\lambda) = \lambda f_0(\lambda)$, $\lambda \in \mathbb{R}$. Then*

$$(2.21) \quad H(t)f(t, \lambda) = \lambda f(t, \lambda), \quad t \in \mathbb{R}, \quad \lambda \in \mathbb{R},$$

has a unique real-valued weak solution $f(t, \lambda)$ with $f(\cdot, \lambda, n) \in C^\infty(\mathbb{R})$, $\lambda \in \mathbb{R}$, $n \in \mathbb{Z}$, satisfying the time evolution

$$(2.22) \quad \dot{f}(t, \lambda) = (B_H(t)f)(t, \lambda) = a(t)f^+(t, \lambda) - a^-(t)f^-(t, \lambda)$$

$$(2.22a) \quad = -2a^-(t)f^-(t, \lambda) + [\lambda + b(t)]f(t, \lambda)$$

$$(2.22b) \quad = 2a(t)f^+(t, \lambda) - [\lambda + b(t)]f(t, \lambda), \quad t \in \mathbb{R}, \quad \lambda \in \mathbb{R},$$

in the weak sense with

$$(2.23) \quad f(0, \lambda) = f_0(\lambda), \quad \lambda \in \mathbb{R}.$$

(Clearly f is smooth w.r.t. λ if f_0 is.) Moreover, if $f(t, \lambda)$, $g(t, \lambda)$ are two weak solutions of (2.21) and (2.22) with initial values $f_0(\lambda)$, $g_0(\lambda)$ respectively, then

$$(2.24) \quad W(f(t, \lambda), g(t, \lambda))(n) = W(f_0(\lambda), g_0(\lambda))(n), \quad \lambda \in \mathbb{R},$$

is independent of $(t, n) \in \mathbb{R} \times \mathbb{Z}$.

Proof. A direct calculation shows that

$$(2.25) \quad F(t, \lambda) := \dot{f}(t, \lambda) - 2a(t)f^+(t, \lambda) + [\lambda + b(t)]f(t, \lambda), \quad (t, \lambda) \in \mathbb{R}^2,$$

satisfies

$$(2.26) \quad H(t)F(t, \lambda) = \lambda F(t, \lambda), \quad (t, \lambda) \in \mathbb{R}^2,$$

in the weak sense whenever $f(t, \lambda)$ satisfies $H(t)f(t, \lambda) = \lambda f(t, \lambda)$ in the weak sense. The requirement (2.22), i.e.,

$$(2.27) \quad F(t, \lambda) = 0, \quad (t, \lambda) \in \mathbb{R}^2,$$

using the equation

$$(2.28) \quad \begin{aligned} f(t, \lambda, n) &= A(t, \lambda)s^0(t, \lambda, n) + B(t, \lambda)c^0(t, \lambda, n) \\ &\quad - \sum_{m=0}^n a(t, 0)^{-1} [c^0(t, \lambda, n)s^0(t, \lambda, m) - c^0(t, \lambda, m)s^0(t, \lambda, n)] \\ &\quad \cdot b(t, m)f(t, \lambda, m), \quad (t, \lambda, n) \in \mathbb{R}^2 \times \mathbb{Z}, \end{aligned}$$

where s^0, c^0 satisfy (2.21) with $b(t) \equiv 0$ and

$$(2.29) \quad s^0(t, \lambda, 0) = c^0(t, \lambda, 1) = 0, \quad s^0(t, \lambda, 1) = c^0(t, \lambda, 0) = 1, \quad (t, \lambda) \in \mathbb{R}^2,$$

is then equivalent to the linear first order system

$$(2.30) \quad \begin{pmatrix} \dot{A} \\ \dot{B} \end{pmatrix} = \begin{pmatrix} -2b_0 + b_1 + \lambda & a_0^{-1}[2\lambda b_0 - 2a_{-1}^2] \\ 2a_0 & b_0 - \lambda \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix},$$

$$a_0(\cdot) := a(\cdot, 0), \quad b_0(\cdot) := b(\cdot, 0).$$

Since

$$(2.31) \quad \begin{aligned} f(t, \lambda, 0) &= B(t, \lambda), \\ f(t, \lambda, 1) &= A(t, \lambda) + a_0(t)^{-1} b_0(t) B(t, \lambda), \end{aligned} \quad (t, \lambda) \in \mathbb{R}^2,$$

an appropriate choice of $A(0, \lambda), B(0, \lambda)$ will indeed guarantee (2.22) and (2.23). The equation (2.24) follows by a straightforward computation of $\partial_t W(f(t, \lambda), g(t, \lambda)) = 0, (t, \lambda) \in \mathbb{R}^2$. \square

Since the existence of positive weak solutions of $H(t)f(t) = 0$ will play a crucial role in Theorem 2.9 we introduce hypothesis

$$(H.2.5) \quad \begin{aligned} (a(t), b(t)) &\in l_{\mathbb{R}}^{\infty}(\mathbb{Z}) \times l_{\mathbb{R}}^{\infty}(\mathbb{Z}), \quad t \in \mathbb{R}, \\ 0 > a(\cdot, n) &\in C^{\infty}(\mathbb{R}), \quad 0 > b(\cdot, n) \in C^{\infty}(\mathbb{R}), \quad n \in \mathbb{Z}, \\ [b(0, n)/a(0, n)] &= o(|n|) \quad \text{as } n \rightarrow \pm\infty \end{aligned}$$

and state

Lemma 2.6. Assume (H.2.5) and $TL(a, b) = 0$, and let $f_0 > 0$ be a weak, positive solution of $H(0)f_0 = 0$. Then, if $f(t)$ is the unique weak solution of

$$(2.32) \quad H(t)f(t) = 0, \quad \dot{f}(t) = B_H(t)f(t), \quad t \in \mathbb{R}, \quad f(0) = f_0 > 0,$$

given by Lemma 2.4, we get

$$(2.33) \quad f(t) > 0 \quad \text{for all } t \in \mathbb{R}.$$

Proof. Equation (2.22a) with $\lambda = 0$ and the transformation

$$(2.34) \quad g(t, n) = f(t, n)e^{-\int_0^t ds \, b(s, n)}, \quad (t, n) \in \mathbb{R} \times \mathbb{Z},$$

yields

$$(2.35) \quad \begin{aligned} \dot{g}(t, n) &= -2a(t, n-1) \exp \left\{ \int_0^t ds [b(s, n-1) - b(s, n)] \right\} g(t, n-1) \\ &:= A(t, n-1)g(t, n-1), \quad (t, n) \in \mathbb{R} \times \mathbb{Z}, \end{aligned}$$

or

$$(2.36) \quad g(t, n) = f_0(n) + \int_0^t ds \, A(s, n-1)g(s, n-1), \quad (t, n) \in \mathbb{R} \times \mathbb{Z}.$$

Next suppose $t \geq 0$. Then, since $A(t, n) > 0, f_0(n) > 0$, iterating (2.36) yields a positive function $g(t, n) > 0$ (implying $f(t, n) > 0$) whenever the

iteration procedure converges absolutely. (In that case it is easily verified that the iterations indeed provide a solution of (2.35).) The m th iteration yields

$$(2.37) \quad \int_0^t dt_1 A(t_1, n-1) \int_0^{t_1} dt_2 A(t_2, n-2) \cdots \int_0^{t_{m-1}} dt_m A(t_m, n-m) f_0(n-m) \\ \leq C(t)^m t^m (m!)^{-1} f_0(n-m), \quad t \geq 0, \quad n \in \mathbb{Z}, \quad m \in \mathbb{N}_0,$$

where we used

$$(2.38) \quad \sup_{0 \leq s \leq t} \|A(s)\|_\infty = C(t), \quad t \geq 0.$$

It remains to bound (2.37). First we note that for some $\gamma > 0$

$$(2.39) \quad f_0(k) \leq \gamma \begin{cases} b(0, 0)b(0, 1) \cdots b(0, k-1) \\ \quad \times [a(0, 0)a(0, 1) \cdots a(0, k-1)]^{-1} f_0(0), & k > 0, \\ b(0, k)b(0, k-1) \cdots b(0, 0) \\ \quad \times [a(0, k)a(0, k-1) \cdots a(0, 0)]^{-1} f_0(0), & k < 0. \end{cases}$$

This follows from (A.9)–(A.11) since $f_0 = \alpha u_- + \beta u_+$ for some $\alpha, \beta \geq 0$ and the fact that (2.39) holds for u_\pm . Indeed, for $k > 0$,

$$(2.40) \quad \begin{aligned} u_+(k) &= [-a(0, 0)] \cdots [-a(0, k-1)] c_{1,+} \cdots c_{k,+} \\ &\leq b(0, 0) \cdots b(0, k-1) [a(0, 0) \cdots a(0, k-1)]^{-1}, \\ u_-(k) &= [-a(0, 0)] \cdots [-a(0, k-1)] c_{0,-} \cdots c_{k-1,-} \\ &\leq b(0, 0) \cdots b(0, k-1) [a(0, 0) \cdots a(0, k-1)]^{-1} \end{aligned}$$

by (A.12) and (A.13). Similarly one treats the case $k < 0$. Thus we get

$$(2.41) \quad \begin{aligned} C(t)^m t^m (m!)^{-1} f_0(n-m) &\leq \gamma f_0(0) C(t)^m t^m (m!)^{-1} \\ &\quad \cdot b(0, n-m) \cdots b(0, n-1) [a(0, n-m) \cdots a(0, n-1)]^{-1} \\ &:= \beta_m(t, n), \quad t \geq 0, \quad m \in \mathbb{N}_0, \quad n \in \mathbb{Z}. \end{aligned}$$

Since by hypothesis (H.2.5),

$$(2.42) \quad \frac{\beta_{m+1}(t, n)}{\beta_m(t, n)} = t C(t) \frac{b(0, n-m-1)}{a(0, n-m-1)} \frac{1}{m+1} \\ = t C(t) o(1) \quad \text{as } m \rightarrow \infty,$$

the iterations converge absolutely for $t \geq 0$. For $t \leq 0$ one uses (2.22b) with $\lambda = 0$ and proceeds analogously. \square

Remark 2.7. The simple example

$$(2.43) \quad \begin{aligned} -a(0, n) &= \begin{cases} (n+1)^{-1}, & n \geq 1, \\ 1, & n \leq 0, \end{cases} & -b(0, n) &= \begin{cases} 1+n^{-2}, & n \geq 1, \\ 2, & n \leq 0, \end{cases} \\ f_0(n) &= \begin{cases} n!, & n \geq 1, \\ 1, & n \leq 0, \end{cases} & H(0)f_0 &= 0, & \left[\frac{b(0, n)}{a(0, n)} \right]_{n \rightarrow \infty} &= O(n), \end{aligned}$$

shows that our condition $o(|n|)$ in (H.2.5) cannot be weakened for the iterations (2.37) to converge for all $t \geq 0$.

Remark 2.8. The connection between Flaschka's and Toda's original variables reads

$$(2.44) \quad \begin{aligned} a(t, n) &:= (\epsilon_n/2)e^{-[Q(t, n+1)-Q(t, n)]/2}, \quad b(t, n) = \dot{Q}(t, n)/2, \\ \epsilon_n &= \pm 1, \quad (t, n) \in \mathbb{R} \times \mathbb{Z}, \\ \ddot{Q} &= e^{-(Q-Q^-)} - e^{-(Q^+-Q)}, \end{aligned}$$

and similarly for the Kac-van Moerbeke system

$$(2.45) \quad \begin{aligned} \alpha(t, n) &:= \epsilon_n(1/2)e^{-R(t, n)/2}, \quad \epsilon_n = \pm 1, \quad (t, n) \in \mathbb{R} \times \mathbb{Z}, \\ \dot{R} &= [e^{-R^-} - e^{-R^+}]/2. \end{aligned}$$

We also note that $T(a, b) = 0$ is invariant under

$$(2.46) \quad a(t, n) \rightarrow \epsilon_n a(t, n), \quad \epsilon_n = \pm 1, \quad b(t, n) \rightarrow b(t, n) + c, \quad c \in \mathbb{R}$$

(independent of (t, n)), and $\text{KM}(\alpha) = 0$ is invariant w.r.t.

$$(2.47) \quad \alpha(t, n) \rightarrow \epsilon_n \alpha(t, n), \quad \epsilon_n = \pm 1.$$

These facts motivate our hypothesis $a < 0$ in Lemma 2.6. The assumption $b < 0$ is then necessary for the existence of $f_0 > 0$ in Lemma 2.6.

Now we are in position to reverse the implication in (2.8).

Theorem 2.9. Assume (a_1, b_1) satisfies (H.2.5) and $\text{TL}(a_1, b_1) = 0$. Suppose $H_1(0) \geq 0$ and let $0 < u_{1, \pm}(\cdot, n) \in C^\infty(\mathbb{R})$, $n \in \mathbb{Z}$, be positive weak solutions of

$$(2.48) \quad H_1(t)u_{1, \pm}(t) = 0, \quad \dot{u}_{1, \pm}(t) = B_{H_1}(t)u_{1, \pm}(t), \quad t \in \mathbb{R}.$$

Define

$$(2.49) \quad u_{1, \sigma}(t, n) := 2^{-1}[1 - \sigma(t)]u_{1, -}(t, n) + 2^{-1}[1 + \sigma(t)]u_{1, +}(t, n),$$

$$(2.50) \quad \phi_\sigma(t, n) := -[-a_1(t, n)u_{1, \sigma}(t, n+1)/u_{1, \sigma}(t, n)]^{1/2},$$

$$(2.51) \quad \psi_\sigma(t, n) := [-a_1(t, n)u_{1, \sigma}(t, n)/u_{1, \sigma}(t, n+1)]^{1/2},$$

$$(2.52) \quad \alpha_\sigma(t, n) := \begin{cases} \phi_\sigma(t, m), & n = 2m, \\ \psi_\sigma(t, m), & n = 2m+1, \end{cases}$$

$$a_{2, \sigma}(t, n) := \phi_\sigma(t, n+1)\psi_\sigma(t, n),$$

$$(2.53) \quad b_{2, \sigma}(t, n) := -\phi_\sigma(t, n)^2 - \psi_\sigma(t, n)^2, \quad (t, n) \in \mathbb{R} \times \mathbb{Z},$$

where $\sigma: \mathbb{R} \rightarrow [-1, 1]$, $\sigma \in C^\infty(\mathbb{R})$. Then $\alpha_\sigma(t) \in l^\infty_{\mathbb{R}}(\mathbb{Z})$, $(a_{2, \sigma}(t), b_{2, \sigma}(t)) \in l^\infty_{\mathbb{R}}(\mathbb{Z}) \times l^\infty_{\mathbb{R}}(\mathbb{Z})$, $t \in \mathbb{R}$, and

$$(2.54) \quad \text{KM}(\alpha_\sigma) = 0, \quad \text{TL}(a_{2, \sigma}, b_{2, \sigma}) = 0 \quad \text{iff} \quad \dot{\sigma} = 0 \text{ or } W(u_{1, -}, u_{1, +}) = 0.$$

Proof. Since

$$(2.55) \quad -b_1(t, n) = \phi_\sigma(t, n)^2 + \psi_\sigma(t, n-1)^2, \quad (t, n) \in \mathbb{R} \times \mathbb{Z},$$

and $-b_1(t) \in l^\infty_{\mathbb{R}}(\mathbb{Z})$, $t \in \mathbb{R}$, the boundedness assertions for α_σ , $a_{2, \sigma}$, $b_{2, \sigma}$ hold. Moreover, one explicitly computes

$$(2.56) \quad \begin{aligned} \text{KM}(\alpha_\sigma)(t, n) &= -4^{-1}\dot{\sigma}(t)\phi_\sigma(t, m)^{-1}u_{1, \sigma}(t, m)^{-2}W(u_{1, -}, u_{1, +}), \\ &\quad t \in \mathbb{R}, \quad n = 2m, \end{aligned}$$

$$(2.57) \quad \text{KM}(\alpha_\sigma)(t, n) = 4^{-1} \dot{\sigma}(t) \psi_\sigma(t, m)^{-1} u_{1,\sigma}(t, m+1)^{-2} W(u_{1,-}, u_{1,+}), \\ t \in \mathbb{R}, \quad n = 2m+1,$$

and similarly

$$(2.58) \quad \text{TL}(a_{2,\sigma}, b_{2,\sigma})(n) = 4^{-1} \dot{\sigma}(t) W(u_{1,-}, u_{1,+}) \\ \cdot \left(u_{1,\sigma}(n+1)^{-2} \left[-\frac{\psi_\sigma(n)}{\phi_\sigma(n+1)} + \frac{\phi_\sigma(n+1)}{\psi_\sigma(n)} \right], 2[u_{1,\sigma}(n)^{-2} - u_{1,\sigma}(n+1)^{-2}] \right), \\ t \in \mathbb{R}, \quad n \in \mathbb{Z}. \quad \square$$

The definitions (2.50)–(2.52) are suggested by $H_1 f_1 = A^* A f_1 = 0$, i.e., by $A f_1 = 0$ (yielding $\psi f_1^+ + \phi f_1 = 0$, cf. (2.12)) and by the requirements $a_1 = \phi \psi$ (cf. (2.7)) and (2.5).

In order to interpret Theorem 2.19 we recall some notions from Appendix A: $H_1(0) \geq 0$ is called critical iff $H_1(0)f = 0$ has a unique positive weak solution $u_{1,0} > 0$ with $u_{1,0}(0) = 1$, and $H_1(0) \geq 0$ is called subcritical iff $H_1(0)f = 0$ has two linearly independent positive weak solutions $u_{1,\pm} > 0$. Since by Theorem A.5, $H_1(0) \geq 0$ has at least one positive weak solution $u_1 > 0$ of $H_1(0)f = 0$, this case distinction is exhaustive, i.e., either $H_1(0) \geq 0$ is critical or subcritical.

Combining Lemmas 2.4 and 2.6 we get

Lemma 2.10. *Let (a_1, b_1) satisfy (H.2.5) and $\text{TL}(a_1, b_1) = 0$ and assume that $H_1(0) \geq 0$. Then $H_1(t)$ is (sub)critical for some $t \in \mathbb{R}$ iff $H_1(t)$ is (sub)critical for all $t \in \mathbb{R}$.*

Remark 2.11. Given a solution (a_1, b_1) of $\text{TL}(a_1, b_1) = 0$, Theorem 2.9 yields a unique solution, say α_0 of $\text{KM}(\alpha_0) = 0$ and of (2.7) iff $H_1(0)$ is critical. In this case we can dispense with the condition that $u_{1,0}$ satisfies $\dot{u}_{1,0} = B_{H_1} u_{1,0}$ since by Lemma 2.4 a certain t -dependent multiple of $u_{1,0}$ certainly will satisfy it. (This t -dependent factor, however, drops out in the definitions (2.50)–(2.52) and hence is irrelevant.) Otherwise, i.e., iff $H_1(0)$ is subcritical, we get a one-parameter family of solutions α_σ of $\text{KM}(\alpha_\sigma) = 0$ and (2.7) indexed by $\sigma \in [-1, 1]$.

Remark 2.12. The “if part” of Theorem 2.9 essentially has been established by [1] (see also [29]). It is our novel “only if part” in Theorem 2.9 that yields a uniqueness or nonuniqueness result for α depending on whether $H_1(0)$ is critical or subcritical.

Remark 2.13. Given α_σ in (2.52), $H_1(t)$ is recovered from α_σ via

$$(2.59) \quad H_1(t) = A_\sigma^*(t) A_\sigma(t), \quad A_\sigma(t) = \psi_\sigma(t) S^+ + \phi_\sigma(t), \\ \phi_\sigma(t, n) = \alpha(t, 2n), \quad \psi_\sigma(t, n) = \alpha_\sigma(t, 2n+1), \quad (t, n) \in \mathbb{R} \times \mathbb{Z}.$$

Moreover, $H_1 f_1 = \lambda f_1$ in the weak sense reads

$$(2.60) \quad a_1(t, n) f_1(t, n+1) + a_1(t, n-1) f_1(t, n-1) - b_1(t, n) f_1(t, n) = \lambda f_1(t, n), \\ a_1(t) = \phi_\sigma(t) \psi_\sigma(t), \quad b_1(t) = -\phi_\sigma(t)^2 - \psi_\sigma^-(t)^2, \quad (t, n) \in \mathbb{R}^2, \quad \sigma \in [-1, 1].$$

Similarly, $H_{2,\sigma}(t) := A_\sigma(t) A_\sigma(t)^*$ implies for weak solutions f_2 of $H_{2,\sigma} f_2 = \lambda f_2$ that

(2.61)

$$a_{2,\sigma}(t, n) f_2(t, n+1) + a_{2,\sigma}(t, n-1) f_2(t, n-1) - b_{2,\sigma}(t) f_2(t, n) = \lambda f_2(t, n),$$

$$a_{2,\sigma}(t) = \phi_\sigma^+(t) \psi_\sigma(t), \quad b_{2,\sigma}(t) = -\phi_\sigma(t)^2 - \psi_\sigma(t)^2, \quad (t, n) \in \mathbb{R}^2, \quad \sigma \in [-1, 1].$$

3. SOLITON SOLUTIONS FOR THE KAC-VAN MOERBEKE SYSTEM

In this section we illustrate the methods of §2 by deriving the soliton solutions of the Kac-van Moerbeke system given the soliton solutions of the Toda lattice.

First we inductively construct the solutions of the Toda lattice using commutation methods [4, 6, 30]. Consider

$$(3.1) \quad H_0^0(z_1) := -(1/2)(S^+ + S^-) + \Omega_1, \quad \Omega_1 = (z_1 + z_1^{-1})/2, \quad 0 < z_1 < 1,$$

with spectrum

$$(3.2) \quad \sigma(H_0^0(z_1)) = [-1 + \Omega_1, 1 + \Omega_1].$$

Writing

$$(3.3) \quad H_0^0(z_1) = A(z_1)^* A(z_1),$$

with

$$(3.4) \quad A(z_1) := \psi_1(z_1) S_+ + \phi_1(z_1),$$

we get

$$(3.5) \quad -1/2 = \psi_1(z_1, n) \phi_1(z_1, n), \quad n \in \mathbb{Z}.$$

We have

$$(3.6) \quad H_0^0(z_1) u_{0,\pm}(z_1) = 0,$$

where

$$(3.7) \quad u_{0,\pm}(z_1, n) = z_1^{\pm n}, \quad n \in \mathbb{Z},$$

and $H_0^0(z_1)$ is subcritical [14]. Define

$$(3.8) \quad u_{0,\mu_1}(z_1) := u_{0,-}(z_1) + \mu_1 u_{0,+}(z_1), \quad \mu_1 > 0.$$

Then $H_0^0(z_1) u_{0,\mu_1}(z_1) = 0$ implies

$$(3.9) \quad A(z_1) u_{0,\mu_1}(z_1) = 0$$

and (3.5), (3.8), and (3.9) yield

$$(3.10) \quad \psi_1(z_1, n) = \left[\frac{z_1^{-n} + \mu_1 z_1^n}{2(z_1^{-n-1} + \mu_1 z_1^{n+1})} \right]^{1/2}, \quad n \in \mathbb{Z},$$

$$(3.11) \quad \phi_1(z_1, n) = - \left[\frac{z_1^{-n-1} + \mu_1 z_1^{n+1}}{2(z_1^{-n} + \mu_1 z_1^n)} \right]^{1/2}, \quad n \in \mathbb{Z}.$$

(The signs of ψ_1 and ϕ_1 are not determined except for the fact that they must be chosen opposite of each other.) Next define

$$(3.12) \quad H_1(z_1) = A(z_1) A(z_1)^* := a_1(z_1) S^+ + a_1^-(z_1) S^- - b_1(z_1).$$

Then

$$(3.13) \quad \begin{aligned} a_1(z_1, n) &= \phi_1(z_1, n+1)\psi_1(z_1, n), \\ b_1(z_1, n) &= -\phi_1(z_1, n)^2 - \psi_1(z_1, n)^2, \quad n \in \mathbb{Z}, \end{aligned}$$

yields

$$(3.14) \quad \begin{aligned} a_1(z_1, n) &= -\frac{(z_1^{-n-2} + \mu_1 z_1^{n+2})^{1/2} (z_1^{-n} + \mu_1 z_1^n)^{1/2}}{2(z_1^{-n-1} + \mu_1 z_1^{n+1})}, \\ b_1(z_1, n) &= -\frac{1}{2} \left[\frac{z_1^{-n-1} + \mu_1 z_1^{n+1}}{z_1^{-n} + \mu_1 z_1^n} + \frac{z_1^{-n} + \mu_1 z_1^n}{z_1^{-n-1} + \mu_1 z_1^{n+1}} \right], \quad n \in \mathbb{Z}. \end{aligned}$$

Moreover, the operator $H_1(z_1)$ is critical with spectrum

$$(3.15) \quad \sigma(H_1(z_1)) = \{0\} \cup [-1 + \Omega_1, 1 + \Omega_1]$$

since the equation

$$(3.16) \quad H_1(z_1)f_1(z_1) = 0, \quad f_1(z_1, 0) = 1,$$

has a unique solution satisfying

$$(3.17) \quad 0 \left(\frac{f_1(z_1, n)}{f_1(z_1, n-1)} \right)_{n \rightarrow \pm\infty} = z_1^{\pm 1}$$

implying $f_1(z_1) \in l^2(\mathbb{Z})$. Next one defines the subcritical operator

$$(3.18) \quad \begin{aligned} H_1^0(z_1, z_2) &:= H_1(z_1) - \Omega_1 + \Omega_2, \\ \Omega_2 &> \Omega_1 > 1, \quad \Omega_j = (z_j + z_j^{-1})/2, \quad 0 < z_j < 1, \quad j = 1, 2, \end{aligned}$$

with spectrum

$$(3.19) \quad \sigma(H_1^0(z_1, z_2)) = \{\Omega_2 - \Omega_1\} \cup [-1 + \Omega_2, 1 + \Omega_2]$$

and repeats the above procedure. We summarize the result in

Theorem 3.1. *The above construction yields operators*

$$(3.20) \quad \begin{aligned} H_N(z_1, \dots, z_N) &= A(z_1, \dots, z_N)A(z_1, \dots, z_N)^* \\ &= a_N(z_1, \dots, z_N)S^+ + a_N^-(z_1, \dots, z_N)S^- - b_N(z_1, \dots, z_N), \end{aligned}$$

where

$$(3.21) \quad A(z_1, \dots, z_N) = \psi_N(z_1, \dots, z_N)S^+ + \phi_N(z_1, \dots, z_N)$$

and

$$(3.22) \quad \begin{aligned} a_N(z_1, \dots, z_N, n) &= -\frac{[W(\rho_1, \dots, \rho_N)(n+2)]^{1/2} [W(\rho_1, \dots, \rho_N)(n)]^{1/2}}{2W(\rho_1, \dots, \rho_N)(n+1)}, \\ b_N(z_1, \dots, z_N, n) &= -\frac{1}{2} \left[\frac{W(\rho_1, \dots, \rho_{N-1})(n+2)W(\rho_1, \dots, \rho_N)(n)}{W(\rho_1, \dots, \rho_{N-1})(n+1)W(\rho_1, \dots, \rho_N)(n+1)} \right. \\ &\quad \left. + \frac{W(\rho_1, \dots, \rho_{N-1})(n)W(\rho_1, \dots, \rho_N)(n+1)}{W(\rho_1, \dots, \rho_{N-1})(n+1)W(\rho_1, \dots, \rho_N)(n)} \right], \quad n \in \mathbb{Z}, \end{aligned}$$

$$(3.23) \quad \begin{aligned} &\psi_N(z_1, \dots, z_N, n) \\ &= \left[\frac{W(\rho_1, \dots, \rho_{N-1})(n+2)W(\rho_1, \dots, \rho_N)(n)}{2W(\rho_1, \dots, \rho_{N-1})(n+1)W(\rho_1, \dots, \rho_N)(n+1)} \right]^{1/2}, \quad n \in \mathbb{Z}, \end{aligned}$$

$$(3.24) \quad \begin{aligned} &\phi_N(z_1, \dots, z_N, n) \\ &= - \left[\frac{W(\rho_1, \dots, \rho_{N-1})(n)W(\rho_1, \dots, \rho_N)(n+1)}{2W(\rho_1, \dots, \rho_{N-1})(n+1)W(\rho_1, \dots, \rho_N)(n)} \right]^{1/2}, \quad n \in \mathbb{Z}. \end{aligned}$$

Here $W(\rho_1, \dots, \rho_N)(n)$ is the discrete Wronskian, i.e.,

$$(3.25) \quad W(\rho_1, \dots, \rho_N)(n) := \det[(S^+)^{j-1}\rho_l(n)]_{j,l=1}^N$$

and

$$(3.26) \quad \rho_j(n) := z_j^{-n} + (-1)^{j-1}\mu_j z_j^n, \quad \mu_j > 0, \quad 1 \leq j \leq N, \quad n \in \mathbb{Z}.$$

Moreover, $H_N(z_1, \dots, z_N)$ is critical with spectrum

$$(3.27) \quad \begin{aligned} \sigma(H_N(z_1, \dots, z_N)) &= \bigcup_{j=1}^N \{\Omega_N - \Omega_j\} \cup [-1 + \Omega_N, 1 + \Omega_N], \\ \Omega_j &= \frac{z_j + z_j^{-1}}{2}, \quad \Omega_N > \Omega_{N-1} > \dots > \Omega_1 > 1, \\ &0 < z_j < 1, \quad 1 \leq j \leq N, \quad N \in \mathbb{N}. \end{aligned}$$

Next we briefly consider an alternative representation for a_N, b_N due to [7].

Lemma 3.2. Define

$$(3.28) \quad \begin{aligned} B_0(t, n) &:= 1, \quad (t, n) \in \mathbb{R} \times \mathbb{Z}, \\ B_N(t, n) &:= \left[\delta_{j,l} + c_j(t)c_l(t) \frac{(z_j z_l)^{n+1}}{1 - z_j z_l} \right]_{j,l=1}^N, \\ &N \in \mathbb{N}, \quad (t, n) \in \mathbb{R} \times \mathbb{Z}, \end{aligned}$$

where

$$(3.29) \quad \begin{aligned} c_j(t) &= c_j(0)e^{\beta_j t}, \quad c_j(0) \in \mathbb{C} \setminus \{0\}, \\ \beta_j &= (z_j - z_j^{-1})/2, \quad 1 \leq j \leq N, \quad t \in \mathbb{R}. \end{aligned}$$

Let

$$(3.30) \quad \eta_j(t, n) := z_j^n e^{z_j t + m_j} + (-1)^{j-1} z_j^{-n} e^{z_j^{-1} t - m_j}, \quad 1 \leq j \leq N, \quad (t, n) \in \mathbb{R} \times \mathbb{Z},$$

with

$$(3.31) \quad c_j(0)^2 = (-1)^j z_j^{N-3} \frac{\prod_{l=1}^N (z_j z_l - 1)}{\prod_{l=1, l \neq j}^N (z_l - z_j)} e^{2m_j}, \quad m_j \in \mathbb{C}, \quad 1 \leq j \leq N.$$

Then the Wronskian

$$W(\eta_1, \dots, \eta_N)(t, n) = \det[(S^+)^{j-1}\eta_l(t, n)] = \det[\partial_t^{j-1}\eta_l(t, n)]$$

satisfies

$$(3.32) \quad \det B_N(t, n) = \left[\prod_{j=1}^N (-1)^{j-1} z_j^{N+n} e^{-z_j^{-1}t+m_j} \right] \left[\prod_{\substack{j,l=1 \\ j \neq l}}^N (z_j^{-1} - z_l^{-1}) \right]^{-1} \\ \cdot W(\eta_1, \dots, \eta_N)(t, n), \quad (t, n) \in \mathbb{R} \times \mathbb{Z},$$

and hence

$$(3.33) \quad \partial_t \ln \det B_N(t, n) = \partial_t \ln W(\eta_1, \dots, \eta_N)(t, n) - \sum_{j=1}^N z_j^{-1}, \\ (t, n) \in \mathbb{R} \times \mathbb{Z}, \quad N \in \mathbb{N}.$$

Proof. Following [8, 10] one defines the $N \times N$ matrices

$$(3.34) \quad P_1 := [\delta_{j,l} p_j], \quad P_2(t) := [\delta_{j,l} q_j(t)], \quad A = [(1 - z_j z_l)^{-1}], \\ D_1(t) := [\delta_{j,l} d_j(t)], \quad D_2 := [\delta_{j,l} z_j^{n+1}], \\ U(t) := [\delta_{j,l} c_j(t)], \quad V := [z_l^{-(j-1)}], \quad W(t) := [\tau_l(t) z_l^{j-1}], \quad t \in \mathbb{R},$$

where

$$(3.35) \quad p_j := \begin{cases} 1, & N = 1, \\ z_j^{N-1} \prod_{\substack{l=1 \\ l \neq j}}^N (z_l - z_j)^{-1}, & N \geq 2, \end{cases} \\ q_j(t) := -z_j^{-1} e^{z_j t + m_j} \prod_{l=1}^N (z_l z_j - 1), \\ d_j(t) := (-1)^{j-1} z_j^n e^{-z_j^{-1}t+m_j}, \\ \tau_j(t) := z_j^{-1} e^{z_j t + m_j}, \quad 1 \leq j \leq N, \quad t \in \mathbb{R}.$$

A tedious calculation then shows that

$$(3.36) \quad U(t) B_N(t, n) U(t)^{-1} = P_2(t) D_1(t) V^{-1} [V D_1(t)^{-1} + W(t) D_2] P_2(t)^{-1}, \\ t \in \mathbb{R},$$

and

$$(3.37) \quad \det[V D_1(t)^{-1} + W(t) D_2](t, n) = W(\eta_1, \dots, \eta_N)(t, n), \\ (t, n) \in \mathbb{R} \times \mathbb{Z},$$

proving (3.32). \square

Inserting the t -dependence (3.29) into (3.22) applying Lemma 3.2 then yields

the N -soliton solutions a_N, b_N of the Toda lattice in the form

$$\begin{aligned}
 a_N(z_1, \dots, z_N, t, n) &= -\frac{[\det B_N(t, n+2)]^{1/2} [\det B_N(t, n)]^{1/2}}{2 \det B_N(t, n+1)} \\
 &= -\frac{[W(\eta_1, \dots, \eta_N)(t, n+2)]^{1/2} [W(\eta_1, \dots, \eta_N)(t, n)]^{1/2}}{2W(\eta_1, \dots, \eta_N)(t, n+1)}, \\
 b_N(z_1, \dots, z_N, t, n) &= -\frac{1}{2} \left[\frac{z_N \det B_{N-1}(t, n+2) \det B_N(t, n)}{\det B_{N-1}(t, n+1) \det B_N(t, n+1)} \right. \\
 &\quad \left. + \frac{\det B_{N-1}(t, n) \det B_N(t, n+1)}{z_N \det B_{N-1}(t, n+1) \det B_N(t, n)} \right] \\
 &= -\frac{1}{2} \left[\frac{W(\eta_1, \dots, \eta_{N-1})(t, n+2) W(\eta_1, \dots, \eta_N)(t, n)}{W(\eta_1, \dots, \eta_{N-1})(t, n+1) W(\eta_1, \dots, \eta_N)(t, n+1)} \right. \\
 &\quad \left. + \frac{W(\eta_1, \dots, \eta_{N-1})(t, n) W(\eta_1, \dots, \eta_N)(t, n+1)}{W(\eta_1, \dots, \eta_{N-1})(t, n+1) W(\eta_1, \dots, \eta_N)(t, n)} \right], \\
 &\quad (t, n) \in \mathbb{R} \times \mathbb{Z},
 \end{aligned}
 \tag{3.38}$$

$$\text{TL}(a_N, b_N) = 0, \quad N \in \mathbb{N}.
 \tag{3.39}$$

Uniqueness of the inverse scattering method for the Toda lattice implies that one may alternatively write b_N as

$$\begin{aligned}
 b_N(z_1, \dots, z_N, t, n) &= -\frac{1}{2}(z_N + z_N^{-1}) + \partial_t \ln \left[\frac{\det B_N(t, n+1)}{\det B_N(t, n)} \right] \\
 &= -\frac{1}{2}(z_N + z_N^{-1}) + \partial_t \ln \left[\frac{W(\eta_1, \dots, \eta_N)(t, n+1)}{W(\eta_1, \dots, \eta_N)(t, n)} \right], \\
 &\quad (t, n) \in \mathbb{R} \times \mathbb{Z}
 \end{aligned}
 \tag{3.40}$$

(the constant $-(z_N + z_N^{-1})/2$ being determined by the asymptotic behavior as $|n| \rightarrow \infty$.) Theorem 2.9 now yields (identifying $H_1(t) = H_N(z_1, \dots, z_N, t)$, $a_1(t) = a_N(z_1, \dots, z_N, t)$, $b_1(t) = b_N(z_1, \dots, z_N, t)$, etc.)

Theorem 3.3. *Since $H_N(z_1, \dots, z_N, t)$ is critical, the $(2N-1)$ -soliton solutions of the Kac-van Moerbeke lattice are given by*

$$\begin{aligned}
 &\alpha_{2N-1}(z_1, \dots, z_N, t, n) \\
 &= \begin{cases} \phi_N(z_1, \dots, z_N, t, m), & n = 2m, \\ \psi_N(z_1, \dots, z_N, t, m), & n = 2m+1, \end{cases} \quad (t, m) \in \mathbb{R} \times \mathbb{Z},
 \end{aligned}
 \tag{3.41}$$

$$\text{KM}(\alpha_{2N-1}) = 0, \quad N \in \mathbb{N},
 \tag{3.42}$$

where

$$\begin{aligned}
 &\phi_N(z_1, \dots, z_N, t, m) \\
 &= -\left[\frac{W(\eta_1, \dots, \eta_{N-1})(t, n) W(\eta_1, \dots, \eta_N)(t, n+1)}{2W(\eta_1, \dots, \eta_{N-1})(t, n+1) W(\eta_1, \dots, \eta_N)(t, n)} \right]^{1/2}, \\
 &\quad (t, m) \in \mathbb{R} \times \mathbb{Z},
 \end{aligned}
 \tag{3.43}$$

$$\begin{aligned}
 &\psi_N(z_1, \dots, z_N, t, m) \\
 &= \left[\frac{W(\eta_1, \dots, \eta_{N-1})(t, n+2) W(\eta_1, \dots, \eta_N)(t, n)}{2W(\eta_1, \dots, \eta_{N-1})(t, n+1) W(\eta_1, \dots, \eta_N)(t, n+1)} \right]^{1/2}, \\
 &\quad (t, m) \in \mathbb{R} \times \mathbb{Z},
 \end{aligned}
 \tag{3.44}$$

and $\eta_j(t, n)$ are given by (3.30). The corresponding Dirac operator $Q_{2N-1}(t)$ (see (2.13)) then has spectrum

$$(3.45) \quad \sigma(Q_{2N-1}(t)) = \bigcup_{j=1}^N \{ \pm |\Omega_N - \Omega_j|^{1/2} \} \cup [-|1 + \Omega_N|^{1/2}, -|-1 + \Omega_N|^{1/2}] \\ \cup [| -1 + \Omega_N|^{1/2}, |1 + \Omega_N|^{1/2}], \quad t \in \mathbb{R}, \quad N \in \mathbb{N}.$$

Example 3.4. In the critical case the 1-soliton solution is given by

$$(3.46) \quad \alpha_1(z_1, t, n) = \begin{cases} - \left[\frac{1 + c_1(t)^2 z_1^{2m+2} (1 - z_1^2)^{-1}}{2z_1 [1 + c_1(t)^2 z_1^{2m} (1 - z_1^2)^{-1}]} \right]^{1/2}, & n = 2m, \\ \left[\frac{z_1 [1 + c_1(t)^2 z_1^{2m} (1 - z_1^2)^{-1}]}{2 [1 + c_1(t)^2 z_1^{2m+2} (1 - z_1^2)^{-1}]} \right]^{1/2}, & n = 2m + 1, \end{cases} \\ c_1(t) = c_1(0) e^{(z_1 - z_1^{-1})t/2}, \quad (t, m) \in \mathbb{R} \times \mathbb{Z},$$

$$(3.47) \quad \text{KM}(\alpha_1) = 0.$$

Remark 3.5. In the subcritical case one can obtain the Kac-van Moerbeke $(2N - 2)$ -soliton solutions by “turning off” the eigenvalue at zero of $H_N(z_1, \dots, z_N, t)$ (rendering it subcritical). This is most conveniently accomplished by using the formulas expressed in $\det B_N(t, n)$ letting $c_N(0) \rightarrow 0$. One then obtains

$$(3.48) \quad \alpha_{2N-2}(z_1, \dots, z_N, t, n) = \begin{cases} \tilde{\phi}_N(z_1, \dots, z_N, t, m), & n = 2m, \\ \tilde{\psi}_N(z_1, \dots, z_N, t, m), & n = 2m + 1, \end{cases} \quad (t, m) \in \mathbb{R} \times \mathbb{Z},$$

$$(3.49) \quad \text{KM}(\alpha_{2N-2}) = 0, \quad N \in \mathbb{N},$$

where

$$(3.50) \quad \tilde{\phi}_N(z_1, \dots, z_N, t, m) = - \left[\frac{\det B_{N-1}(t, m) \det \tilde{B}_{N-1}(t, m+1)}{2z_N \det B_{N-1}(t, m+1) \det \tilde{B}_{N-1}(t, m)} \right]^{1/2} \\ = - \left[\frac{W(\eta_1, \dots, \eta_{N-1})(t, m) W(\tilde{\eta}_1, \dots, \tilde{\eta}_{N-1})(t, m+1)}{2z_N W(\eta_1, \dots, \eta_{N-1})(t, m+1) W(\tilde{\eta}_1, \dots, \tilde{\eta}_{N-1})(t, m)} \right]^{1/2}, \\ (t, m) \in \mathbb{R} \times \mathbb{Z},$$

$$(3.51) \quad \tilde{\psi}_N(z_1, \dots, z_N, t, m) = \left[\frac{z_N \det B_{N-1}(t, m+2) \det \tilde{B}_{N-1}(t, m)}{2 \det B_{N-1}(t, m+1) \det \tilde{B}_{N-1}(t, m+1)} \right]^{1/2} \\ = \left[\frac{z_N W(\eta_1, \dots, \eta_{N-1})(t, m) W(\tilde{\eta}_1, \dots, \tilde{\eta}_{N-1})(t, m)}{2 W(\eta_1, \dots, \eta_{N-1})(t, m+1) W(\tilde{\eta}_1, \dots, \tilde{\eta}_{N-1})(t, m+1)} \right]^{1/2}, \\ (t, m) \in \mathbb{R} \times \mathbb{Z}.$$

Here $\tilde{B}_{N-1}(t, n)$ is the matrix

$$(3.52) \quad \begin{aligned} \tilde{B}_0(t, n) &:= 1, \quad (t, n) \in \mathbb{R} \times \mathbb{Z}, \\ \tilde{B}_{N-1}(t, n) &:= \left[\delta_{j,l} + \frac{\tilde{c}_j(t)\tilde{c}_l(t)}{1 - z_j z_l} (z_j z_l)^{n+1} \right]_{j,l=1}^{N-1}, \quad N \geq 2, \quad (t, n) \in \mathbb{R} \times \mathbb{Z}, \end{aligned}$$

with

$$(3.53) \quad \begin{aligned} \tilde{c}_j(t) &= \tilde{c}_j(0) e^{(z_j - z_j^{-1})t/2}, \quad \tilde{c}_j(0)^2 = z_j \frac{z_j z_N - 1}{z_N - z_j} c_j(0)^2, \\ & \quad j = 1, \dots, N-1, \quad t \in \mathbb{R}, \end{aligned}$$

$$(3.54) \quad \begin{aligned} \tilde{\eta}_j(t, n) &= z_j^n e^{z_j t + \tilde{m}_j} + (-1)^{j-1} z_j^{-n} e^{z_j^{-1} t - \tilde{m}_j}, \\ & \quad j = 1, \dots, N-1, \quad (t, n) \in \mathbb{R} \times \mathbb{Z}, \end{aligned}$$

$$(3.55) \quad \tilde{m}_j = m_j + \frac{1}{2} \ln \left[\frac{(z_j z_{N-1}) z_j}{z_N - z_j} \right], \quad j = 1, \dots, N-1.$$

Remark 3.6. The approach presented in this section relies on “single-commutation” [4] and therefore introduces all eigenvalues below the essential spectrum. This limitation can easily be removed by using “double-commutation” instead [30].

Remark 3.7. The first one to give the N -soliton solution of the Toda lattice appears to be Hirota [18]. The Toda N -soliton solutions expressed in terms of $\det B(t, n)$ are due to Flaschka [7] (see also [29]). Marchenko [22] gives the $2N$ -soliton solutions for both the TL and the KM-systems in terms of Wronskians in the subcritical case using an entirely different approach from the one advocated here. Although he does not list the soliton formulas corresponding to the critical case, they can be obtained from his expressions via an appropriate limiting procedure similar to the one employed in Remark 3.5.

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APPENDIX A. ON NONNEGATIVE, SECOND ORDER, FINITE DIFFERENCE OPERATORS

We summarize the necessary facts on positive weak solutions of nonnegative, second order, finite difference operators on \mathbb{Z} needed in this manuscript. For a novel approach and detailed proofs of the topics presented below we refer to [14]. Basic results on finite difference operators can be found, e.g., in [2, 3, 15, 17].

For simplicity we shall assume hypothesis

$$(H.A.1) \quad \begin{aligned} a &= \{a(n)\}_{n \in \mathbb{Z}} \in l^\infty(\mathbb{Z}), \quad b = \{b(n)\}_{n \in \mathbb{Z}} \in l^\infty(\mathbb{Z}), \\ & \quad a(n) < 0, \quad b(n) < 0, \quad n \in \mathbb{Z}. \end{aligned}$$

throughout this appendix (the assumption $a, b \in l^\infty(\mathbb{Z})$ is removed in [14]) and define the operator

$$(A.1) \quad H = aS^+ + a^-S^- - b$$

on $l^2(\mathbb{Z})$. Associated with H we introduce the matrices $M_{m,n}$ defined by

$$(A.2) \quad M_{m,n} := \begin{pmatrix} -b(m) & a(m) & 0 & \dots & 0 \\ a(m) & -b(m+1) & a(m+1) & & \vdots \\ 0 & & & & 0 \\ \vdots & & & & \\ 0 & \dots & a(n-2) & -b(n-1) & a(n-1) \\ & & 0 & a(n-1) & -b(n) \end{pmatrix}, \quad m < n,$$

and the determinants

$$(A.3) \quad d_{m,n} := \det[M_{m,n}], \quad m < n.$$

Lemma A.2. Assume (H.A.1).

- (i) Let $m < n$. Then $M_{m,n} \geq 0$ implies $d_{m,n-1} > 0$.
- (ii) $H \geq 0$ iff $d_{m,n} > 0$ for any $m < n$.
- (iii) Let $H \geq 0$. Then for all $m < k < n$,

$$(A.4) \quad -b(k)^{-1} < d_{k+1,n}(d_{k,n})^{-1} < -b(k-1)a(k-1)^{-2},$$

$$(A.5) \quad -b(k)^{-1} < d_{m,k-1}(d_{m,k})^{-1} < -b(k+1)a(k)^{-2}.$$

- (iv) For any $k \in \mathbb{Z}$, the map

$$(A.6) \quad n \rightarrow d_{k+1,n}(d_{k,n})^{-1}$$

is increasing for $n > k$ and the map

$$(A.7) \quad m \rightarrow d_{m,k-1}(d_{m,k})^{-1}$$

is decreasing for $m < k$.

Because of Lemma A.2(iv) we define in the case $H \geq 0$

$$(A.8) \quad c(k)_+ := \lim_{n \rightarrow \infty} d_{k+1,n}(d_{k,n})^{-1}, \quad c(k)_- := \lim_{m \rightarrow -\infty} d_{m,k-1}(d_{m,k})^{-1}$$

and the following positive functions on $\mathbb{Z} \times \mathbb{Z}$:

$$(A.9) \quad u(l, k) := \begin{cases} 1, & l = k, \\ [-a(l)] \cdots [-a(k-1)]c(l+1)_+ \cdots c(k)_+, & l < k, \\ [-a(k)] \cdots [-a(l-1)]c(k)_- \cdots c(l-1)_-, & l > k, \end{cases}$$

$$(A.10) \quad u_+(k) := \begin{cases} 1, & k = 0, \\ u(0, k), & k > 0, \\ u(k, 0)^{-1}, & k < 0, \end{cases}$$

$$(A.11) \quad u_-(k) := \begin{cases} 1, & k = 0, \\ u(k, 0)^{-1}, & k > 0, \\ u(0, k), & k < 0. \end{cases}$$

We have

Lemma A.3. Assume (H.A.1) and $H \geq 0$. Then

(i)

$$(A.12) \quad -b(k)^{-1} \leq c(k)_+ \leq -b(k-1)a(k-1)^{-2},$$

$$(A.13) \quad -b(k)^{-1} \leq c(k)_- \leq -b(k+1)a(k)^{-2}.$$

(ii) $k \rightarrow u(l, k)$ is the minimal weak positive solution of $Hf = 0$ with $f(l) = 1$ for $k \geq l$ and $k \leq l$ separately (i.e., if $v > 0$ is a weak positive solution of $Hv = 0$ with $v(l) = 1$ in $[l, \infty) \cap \mathbb{Z}$ then $v(k) \geq u(l, k)$ for all $k \geq l$ and similarly on $(-\infty, l] \cap \mathbb{Z}$).

(iii) u_{\pm} are weak positive solutions of $Hf = 0$ on \mathbb{Z} .

(iv) Let $u_j > 0$, $j = 1, 2$, be positive solutions of $Hf = 0$ with $u_1(0) = u_2(0)$. If for some $k_0 > 0$, $u_1(k_0) \underset{=}{>} u_2(k_0)$ then $u_1(k) \underset{=}{>} u_2(k)$ for all $k > 0$. Similarly, if for some $k_0 < 0$, $u_1(k_0) \geq u_2(k_0)$ then $u_1(k) \underset{=}{>} u_2(k)$ for all $k < 0$.

(v) For any $m \leq l \leq k$ we have

$$(A.14) \quad u(m, k) = u(m, l)u(l, k), \quad u(k, m) = u(k, l)u(l, m).$$

(vi) For any $l, m \in \mathbb{Z}$,

$$(A.15) \quad u(m, l)u(l, m) \leq 1.$$

(vii)

$$(A.16) \quad \sum_{k=n}^{\infty} [-a(k)u_+(k)u_+(k+1)]^{-1} = \infty$$

$$= \sum_{k=n}^{-\infty} [-a(k)u_-(k)u_-(k+1)]^{-1}, \quad n \in \mathbb{Z}.$$

Remark A.4. The weak positive solutions u_{\pm} are the analogs of principal solutions near $\pm\infty$ of nonoscillatory Sturm-Liouville operators on \mathbb{R} (see, e.g., [13, 16]). They are also called recessive in the literature (see, e.g., [27]). One can show [27] that if $H \geq 0$ and $u_{\pm} > 0$ are principal solutions of $Hf = 0$ near $\pm\infty$

$$(A.17) \quad \hat{u}_+(n) := u_+(n) \sum_{j=m}^{n-1} [-a(j)u_+(j)u_+(j+1)]^{-1}, \quad n \geq m+1,$$

$$(A.18) \quad \hat{u}_-(n) := u_-(n) \sum_{j=n}^m [-a(j)u_-(j)u_-(j+1)]^{-1}, \quad n \leq m,$$

are nonprincipal (or dominant) solutions of $Hf = 0$. Conversely, if $\hat{u}_{\pm} > 0$ are nonprincipal solutions of $Hf = 0$ near $\pm\infty$ then

$$(A.19) \quad u_+(n) := \hat{u}_+(n) \sum_{j=n}^{\infty} [-a(j)\hat{u}_+(j)\hat{u}_+(j+1)]^{-1}, \quad n \in \mathbb{Z},$$

$$(A.20) \quad u_-(n) := \hat{u}_-(n) \sum_{j=n-1}^{-\infty} [-a(j)\hat{u}_-(j)\hat{u}_-(j+1)]^{-1}, \quad n \in \mathbb{Z},$$

are principal solutions of $Hf = 0$ near $\pm\infty$. In any case, nonprincipal solutions $\hat{u}_{\pm} > 0$ near $\pm\infty$ are characterized by [27]

$$(A.21) \quad \sum_{j=n}^{\pm\infty} [-a(n)\hat{u}_{\pm}(j)\hat{u}_{\pm}(j+1)]^{-1} < \infty, \quad n \in \mathbb{Z}.$$

We also recall

Theorem A.5. Assume (H.A.1). Then $H \geq 0$ iff there exists a weak positive solution $u > 0$ of $Hf = 0$.

Moreover, we have

Theorem A.6 [12]. Let $H \geq 0$. Then conditions (i)–(vi) are equivalent.

(i) There exists a unique weak positive solution $u > 0$ of $Hf = 0$ with $u(0) = 1$.

(ii) $u_+ = u_-$.

(iii) For some $k, l \in \mathbb{Z}$, $k \neq l$,

$$u(k, l)u(l, k) = 1.$$

(iv) For any $k, l \in \mathbb{Z}$,

$$u(k, l)u(l, k) = 1.$$

(v) For some $k \in \mathbb{Z}$,

$$a(k)^2 c(k)_- c(k+1)_+ = 1.$$

(vi) For any $k \in \mathbb{Z}$,

$$a(k)^2 c(k)_- c(k+1)_+ = 1.$$

Because of Theorems A.5 and A.6 we introduce in analogy to Sturm-Liouville and general Schrödinger-type operators [13, 25, 28] the definition

Definition A.7. Let $H \geq 0$. Then H is called *critical* iff H satisfies one of the conditions (i)–(vi) in Theorem A.6. Otherwise H is called *subcritical*.

In particular, $H \geq 0$ is critical (or subcritical) iff $Hf = 0$ has a unique (or two linearly independent) positive solution(s) up to multiples of constants.

In the following we write $H(a, b)$ instead of H in order to exhibit the (a, b) -dependence of H .

Theorem A.8 [14]. Let $H(a, b) \geq 0$. Then the following conditions (i)–(v) are equivalent.

(i) $H(a, b)$ is critical.

(ii) For some $k \in \mathbb{Z}$ and any $\epsilon > 0$, $H(a - \epsilon\delta_k, b) \not\geq 0$.

(iii) For any $k \in \mathbb{Z}$ and any $\epsilon > 0$, $H(a - \epsilon\delta_k, b) \not\geq 0$.

(iv) For some $k \in \mathbb{Z}$ and any $\epsilon > 0$, $H(a, b + \epsilon\delta_k) \not\geq 0$.

(v) For any $k \in \mathbb{Z}$ and any $\epsilon > 0$, $H(a, b + \epsilon\delta_k) \not\geq 0$,

where

$$\delta_k(n) = \begin{cases} 1, & k = n, \\ 0, & k \neq n. \end{cases}$$

Let $l_0(\mathbb{Z})$ denote the set of complex-valued sequences $g = \{g(n)\}_{n \in \mathbb{Z}}$ with only finitely many $g(n) \neq 0$. Then we get

Corollary A.9. Let $H(a, b) \geq 0$. Then $H(a, b)$ is subcritical iff for any real valued $f, g \in l_0(\mathbb{Z})$ there exists an $\epsilon > 0$ such that $H(a + \epsilon f, b + \epsilon g) \geq 0$.

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